

Quadratic BSDEs with rough drivers and L^2 -terminal condition

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Abstract

In this paper, we study the existence and uniqueness of solutions to quadratic Backward Stochastic Differential Equations (QBSDEs for short) with rough driver and square integrable terminal condition. The main idea consists in using both Doss-Sussman and Zvonkin type transformations. As an application we study connection between QBSDEs and quadratic PDEs with rough drivers. We also obtain Backward Doubly SDEs and QBSDEs driven by Fractional Brownian with Hurst parameter greater than $\frac{1}{4}$ as particular cases of our QBSDEs with rough drivers. A probabilistic representation of a class of rough quadratic PDE is also proved.

Keywords: Quadratic backward stochastic differential equations, rough paths, Doss-Sussman transformation, Zvonkin type transformation, Itô-Krylov formula, quadratic rough PDE.

1 Introduction

On a Brownian motion setting a Backward stochastic differential equations (BSDE) with terminal variable ξ at time horizon T and generator g is solved by a pair of processes (Y, Z) on the interval $[0, T]$ satisfying

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T,$$

where $(W_t)_{0 \leq t \leq T}$ is a standard d -dimensional Brownian motion. Due to their interesting applications in control theory and in partial differential equations (PDE) they have been extensively studied since the first paper of Pardoux and Peng [21] where they proved that there exists a unique solution to this equation when the terminal condition ξ and the coefficient g satisfy smooth square integrability assumptions and if $g(t, \omega, y, z)$ is Lipschitz in (y, z) uniformly in (t, ω) . Since then, several contributions have been done for relaxing those assumptions. Kobylanski [14] studied a BSDE when the generator g is continuous and has a quadratic growth in z and the terminal condition is bounded. Since then, there were many works on QBSDE. We notice that, all established results on QBSDE require the terminal condition to be bounded or of finite exponential moments, see among others [4], [16]. Recently, Bahlali *et al.* [2] have

studied one-dimensional QBSDE with square integrable terminal value. More precisely they established existence and uniqueness of square integrable solutions for a class of QBSDE when the generator g is dominated by a generator of the form $f(y)|z|^2$ where f is measurable and integrable over \mathbb{R} .

Motivated by rough path PDE (see for instance [6], [7], [10] and [13]). Diehl and Friz [11] considered

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds + \int_t^T G(Y_s) d\eta_s - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \quad (1.1)$$

where $G = (G_1, \dots, G_d)$ is a vector field in \mathbb{R} , with $G_k : \mathbb{R} \rightarrow \mathbb{R}$, $k = 1, \dots, d$. η is a general geometric rough path, which by definition means that there exists a sequence of smooth paths (η^n) converging to η in p -variation rough path metric, $p \geq 1$. When ξ is bounded and the function g is of quadratic growth in z , they used stability theory developed in [14] and proved existence and uniqueness of solutions to QBSDE (1.1). The theory of rough paths has been the subject of several papers and lecture notes. We refer for instance to ([12], [15], [17], [18] and [19]) for interesting research works in this domain.

The first aim of this paper is the study of the QBSDE (1.1) when the terminal data ξ is square integrable and the generator g satisfy a quadratic growth condition in z to be specified the assumptions below. Under some hypothesis weaker than those in [11] we prove existence of a solution to the QBSDE (1.1). In some particular cases we establish also uniqueness. Indeed the boundedness of the random variable ξ , the continuity of the generator g and the linear growth of its partial derivatives are not needed for the existence and uniqueness of solutions. The main tools is to use Doss-Sussman and Zvonkin transformations.

The second aim consists in giving probabilistic representations of some quadratic PDE with rough drivers in the Markovian framework. Moreover some particular rough path are presented to deduce from our first result existence and/or uniqueness for backward doubly stochastic differential equations (BDSDE) and QBSDE driven by fractional Brownian motion (fBm) with Hurst parameter greater than $\frac{1}{4}$.

Our contribution presents clearly as generalization of the main results of Bahlali *et al.* [2] and [3] to QBSDE with rough drivers and extend the results of [11] to merely square integrable terminal data and measurable and integrable generator.

The rest of this paper is organized as follows. Section 2 is devoted to the notations, definitions and the main assumptions on the data. In Section 3 we state and prove our main result concerning the existence of QBSDE with rough driver and square integrable terminal condition ξ . In Section 4 we study the solvability of a class of QBSDE with rough driver by using a Zvonkin transformation. In Section 5 we restrict our selves to a Markovian setting and study a class of PDE driven by rough path. In Section 6 we establish the connection to BDSDE. In this context, firstly η is replaced by a standard Brownian motion. Secondly we investigate the case when η is replaced by a fBm with Hurst parameter $H > \frac{1}{4}$.

2 Notations, definitions and assumptions

We fix once and for all a time interval $[0, T]$ and a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ which carries a d -dimensional Brownian motion W . For a vector x we denote the euclidean norm as usual by $|x|$. Let $(\mathcal{F}_t)_{0 \leq t \leq T}$ be the usual filtration of W . In order to simplify the notations, we sometimes write Y for the process $(Y_t)_{0 \leq t \leq T}$. Denote by \mathcal{M}^2 the space of predictable

processes Z in \mathbb{R}^d such that

$$\|Z\|^2 := \mathbb{E} \left[\int_0^T |Z_s|^2 ds \right] < \infty.$$

Denote by \mathcal{S}^2 the space of \mathbb{R} -valued predictable processes Y such that

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t|^2 \right] < \infty.$$

$$\mathcal{W}_{p,loc}^2(\mathbb{R}) := \{u : \mathbb{R} \rightarrow \mathbb{R} : u, u', u'' \in L_{loc}^p(\mathbb{R})\}.$$

$$\mathcal{L}^2 := \left\{ Z, \mathcal{F}_t\text{-adapted such that } \int_0^T |Z_s|^2 ds < \infty \text{ a.s.} \right\}.$$

For a matrix M we denote by $|M|$, depending on the situation, either the 1-norm, the 2-norm or the ∞ -norm. For $p \geq 1$, $G^{[p]}(\mathbb{R}^d)$ is the free step- $[p]$ nilpotent group over the space \mathbb{R}^d , realized as subset of $\mathbb{R} \oplus \mathbb{R}^d \oplus (\mathbb{R}^d)^{\otimes 2} \oplus \dots \oplus (\mathbb{R}^d)^{\otimes [p]}$, equipped with Carnot-Caratheodory norm as defined in [11]. Here $[p]$ denotes the largest integer not larger than p . $\mathcal{C}_0^{p-var}([0, T], G^{[p]}(\mathbb{R}^d))$ is the set of geometric p -variation rough paths

$$\eta : [0, T] \longrightarrow G^{[p]}(\mathbb{R}^d),$$

starting from 0. For more and technical details on geometric rough path spaces, we refer to [12] Section 9, but they are not necessary for the understanding of this paper. Let V be a vector space and $(\mathcal{X}, |\cdot|)$ a Banach vector space on \mathbb{R}^d . For $\gamma > 0$, we set $[\gamma] := \gamma - 1$, if $\gamma \in \mathbb{N}$ and $[\gamma] := \lfloor \gamma \rfloor$, the integer part of γ , if $\gamma \notin \mathbb{N}$. For $j \in \mathbb{N}$, $d^j f$ denotes the derivative of order j of the differentiable function f . For $x \in \mathbb{R}^n$, we denote $D := D_x = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$ and $D^2 := D_{xx} = (\frac{\partial^2}{\partial x_i \partial x_j})_{i,j=1}^n$. Referring to [12] we recall the following definition

Definition 1. For $\gamma > 0$, f is a γ -Lipschitz function on V if

1. $f : V \longrightarrow \mathcal{X}$ is $[\gamma]$ -times differentiable.
2. $d^j f$ is bounded by K , for all $j = 0, \dots, [\gamma]$.
3. $d^{[\gamma]} f$ is $(\gamma - [\gamma])$ -Hölder, with Hölder constant K , i.e.

$$\text{for all } x \neq y \in V, \quad \frac{|d^{[\gamma]} f(x) - d^{[\gamma]} f(y)|}{|x - y|^{\gamma - [\gamma]}} \leq K. \quad (2.2)$$

We denote by $Lip^\gamma(V)$ the set of γ -Lipschitz functions on V . The smallest constant K for which the inequality (2.2) is satisfied is called the Lipschitz norm of f and is denoted by $\|f\|_{Lip^\gamma(V)}$.

In what follows, we will refer to equation (1.1) as: BSDE(ξ, g, G, η).

Definition 2. We call (Y, Z) a solution of the BSDE(ξ, g, G, η) if

- (i) $(Y, Z) \in \mathcal{S}^2 \times \mathcal{M}^2$.
- (ii) For each $t \in [0, T]$, (Y_t, Z_t) satisfies (1.1).

Assumptions:

(H1) The function g is continuous in (y, z) , for a.e. (t, ω) .

(H2) \mathbb{P} -a.s. for $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^m$,

$$|g(t, y, z)| \leq a + b|y| + c|z| + f(|y|)|z|^2,$$

where a, b and c are positive real numbers (which may change from line to line), and f a positive integrable function.

(H3) For given real numbers $\gamma > p \geq 1$ and $C_G > 0$, we have

$$|G|_{Lip^{\gamma+2}(\mathbb{R})} := \sup_{i=1, \dots, d} |G_i|_{Lip^{\gamma+2}(\mathbb{R})} \leq C_G.$$

As a consequence of Theorem 3.1 in [2], we get the following Proposition.

Proposition 1. *Let G be Lipschitz on \mathbb{R} , $\xi \in L^2(\mathcal{F}_T)$ and η a given smooth path. Under the assumptions (H1)–(H2) the QBSDE*

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds + \int_t^T G(Y_s) d\eta_s - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \quad (2.3)$$

has at least one solution.

Proof. The path η is smooth, then we can rewrite the QBSDE as

$$\begin{aligned} Y_t &= \xi + \int_t^T g(s, Y_s, Z_s) ds + \int_t^T G(Y_s) \dot{\eta}_s ds - \int_t^T Z_s dW_s \\ &= \xi + \int_t^T (g(s, Y_s, Z_s) + G(Y_s) \dot{\eta}_s) ds - \int_t^T Z_s dW_s. \end{aligned}$$

Since

$$|g(s, y, z) + G(y) \dot{\eta}_s| \leq |g(s, y, z)| + |G(y) \dot{\eta}_s|,$$

by (H2) and the Lipschitz property of G we get

$$|g(s, y, z) + G(y) \dot{\eta}_s| \leq a + b|y| + c|z| + f(|y|)|z|^2 + |G(0)| |\dot{\eta}_s| + C|y| |\dot{\eta}_s|,$$

where C is the Lipschitz constant of G . Without loss of generality, we may assume $\dot{\eta}$ to be bounded then we obtain

$$|g(s, y, z) + G(y) \dot{\eta}_s| \leq a + b|y| + c|z| + f(|y|)|z|^2.$$

The rest of the proof follows from Theorem 3.1 in [2].

3 Main results

3.1 Doss-Sussman transformation

The first main results of this paper is the existence of solutions to the rough quadratic equation BSDE(ξ, g, G, η) when $\eta \in \mathcal{C}_0^{p-var}([0, T], G^{[p]}(\mathbb{R}^d))$ and ξ is square integrable.

Theorem 1. *Let $\gamma > p \geq 1$, $\eta \in \mathcal{C}_0^{p-var}([0, T], G^{[p]}(\mathbb{R}^d))$ and $\xi \in L^2(\mathcal{F}_T)$. Assume (H1)–(H3) hold. Then the quadratic BSDE(ξ, g, G, η) has at least one solution.*

To prove this theorem we need some technical Lemmas.

Let ϕ be the flow solution of the following Ordinary Differential Equations (ODE)

$$\phi(t, y) = y + \int_t^T \sum_{k=1}^d G_k(\phi(s, y)) d\eta_s^k, \quad (3.4)$$

with y -inverse ϕ^{-1} is given by

$$\phi^{-1}(t, y) = y - \int_t^T \sum_{k=1}^d \partial_y \phi^{-1}(s, y) G_k(y) d\eta_s^k.$$

The following lemma gives a way to prove and construct a solution to QBSDE (2.3) using Doss-Sussmann transformation to remove the term containing the rough path.

Lemma 1. *Let us given a Lipschitz function G on \mathbb{R} , a square integrable random variable ξ and a smooth path η . Let ϕ be the flow defined in (3.4). A couple (Y, Z) is a solution of the QBSDE (2.3) if and only if the process (\tilde{Y}, \tilde{Z}) defined as*

$$\tilde{Y}_t := \phi^{-1}(t, Y_t), \quad \tilde{Z}_t := \frac{1}{\partial_y \phi(t, \tilde{Y}_t)} Z_t,$$

satisfies the BSDE

$$\tilde{Y}_t = \xi + \int_t^T \tilde{g}(s, \tilde{Y}_s, \tilde{Z}_s) ds - \int_t^T \tilde{Z}_s dW_s, \quad (3.5)$$

where (throughout, ϕ and all its derivatives will always be evaluated at (t, \tilde{y}))

$$\tilde{g}(t, \tilde{y}, \tilde{z}) := \frac{1}{\partial_y \phi} (g(t, \phi, \partial_y \phi \tilde{z}) + \frac{1}{2} \partial_y^2 \phi |\tilde{z}|^2).$$

Proof. Suppose that the process (Y, Z) is a solution of the QBSDE (2.3). Denoting by $\psi := \phi^{-1}$ and $\theta_s = (s, Y_s)$, we have by Itô's formula

$$\begin{aligned} \psi(\theta_t) &= \xi - \int_t^T \sum_{k=1}^d \partial_y \psi(\theta_s) G^k(Y_s) \dot{\eta}_s^k ds + \int_t^T \partial_y \psi(\theta_s) g(s, Y_s, Z_s) ds \\ &\quad + \int_t^T \sum_{k=1}^d \partial_y \psi(\theta_s) G^k(Y_s) \dot{\eta}_s^k ds - \int_t^T \partial_y \psi(\theta_s) Z_s dW_s \\ &\quad - \frac{1}{2} \int_t^T \partial_y^2 \psi(\theta_s) |Z_s|^2 ds \\ &= \xi + \int_t^T \left(\partial_y \psi(\theta_s) g(s, Y_s, Z_s) - \frac{1}{2} \partial_y^2 \psi(\theta_s) |Z_s|^2 \right) ds \\ &\quad - \int_t^T \langle \partial_y \psi(\theta_s) Z_s, dW_s \rangle. \end{aligned}$$

Now, by deriving the identity $\psi(t, \phi(t, \tilde{y})) = \tilde{y}$ we get

$$1 = \partial_y \psi \partial_y \phi, \quad \text{and} \quad 0 = \partial_y^2 \psi (\partial_y \phi)^2 + \partial_y \psi \partial_y^2 \phi.$$

And hence,

$$\partial_y \psi = \frac{1}{\partial_y \phi} \quad \text{and} \quad \partial_y^2 \psi = -\frac{\partial_y \psi \partial_y^2 \phi}{(\partial_y \phi)^2} = -\frac{\partial_y^2 \phi}{(\partial_y \phi)^3}.$$

Define

$$\tilde{Y}_t := \phi^{-1}(t, Y_t), \quad \tilde{Z}_t := \frac{1}{\partial_y \phi(t, \tilde{Y}_t)} Z_t,$$

and $(\psi$ and its derivatives are always evaluated at $(t, \phi(t, \tilde{y}))$, ϕ and its derivatives are always evaluated at $(t, \tilde{y}))$

$$\tilde{g}(t, \tilde{y}, \tilde{z}) := \frac{1}{\partial_y \phi} (g(t, \phi, \partial_y \phi \tilde{z}) + \frac{1}{2} \partial_y^2 \phi |\tilde{z}|^2).$$

Thanks to non-explosion condition (c.f. Condition 4.3 in [12], Section 4, p. 69) for ODEs we can show that the flow ϕ , its derivative $\partial_y \phi$ and its inverse ϕ^{-1} are bounded. This also combined with a localization argument yield to the boundedness of the map $\frac{1}{\partial_y \phi}$. Then we deduce that $(\tilde{Y}, \tilde{Z}) \in \mathcal{S}^2 \times \mathcal{M}^2$. We therefore obtain

$$\tilde{Y}_t := \xi + \int_t^T \tilde{g}(s, \tilde{Y}_s, \tilde{Z}_s) ds - \int_t^T \tilde{Z}_s dW_s. \quad (3.6)$$

To establish the converse, we reverse the transformation and apply the Itô's formula to the process $Y_t = \phi(t, \tilde{Y}_t)$.

For ease of notations, we refer to equation (3.6) with data $(\xi, \tilde{g}, 0, 0)$ as BSDE $(\xi, \tilde{g}, 0, 0)$.

Remark 1. *If one takes $g(t, y, z) = f(y)z^2$, where f is an integrable function, then the uniqueness of solutions (Y, Z) is deduced from the uniqueness of solutions (\tilde{Y}, \tilde{Z}) . A technical proof will be given in Corollary 1 when we deal with the rough path η .*

Now, we consider $\eta \in \mathcal{C}_0^{p-var}([0, T], G^{[p]}(\mathbb{R}^d))$ and record properties (H1) and (H2) for the induced function \tilde{g} of the previous Lemma.

Lemma 2. *Let $\gamma > p \geq 1$ and $\eta \in \mathcal{C}_0^{p-var}([0, T], G^{[p]}(\mathbb{R}^d))$. Assume (H1)-(H3) hold. Let Φ be the flow of the Rough Differential Equations (RDE)*

$$\Phi(t, y) = y + \int_t^T \sum_{k=1}^d G_k(\Phi(s, y)) d\eta_s^k. \quad (3.7)$$

Then the function

$$\tilde{g}(t, \tilde{y}, \tilde{z}) := \frac{1}{\partial_y \Phi} (g(t, \Phi, \partial_y \Phi \tilde{z}) + \frac{1}{2} \partial_y^2 \Phi |\tilde{z}|^2)$$

satisfies the following properties:

- (i) \tilde{g} is continuous.
- (ii) There exists positive real numbers $\tilde{a}, \tilde{b}, \tilde{c}$ and a positive integrable function \tilde{f} such that

$$|\tilde{g}(t, \tilde{y}, \tilde{z})| \leq \tilde{a} + \tilde{b}|\tilde{y}| + \tilde{c}|\tilde{z}| + \tilde{f}(|\tilde{y}|)|\tilde{z}|^2.$$

Proof. (i). Note that the continuity of the function \tilde{g} follows from the continuity of the flow Φ and the continuity of the function g .

(ii). By the property (H2) we have

$$\begin{aligned} |\tilde{g}(t, \tilde{y}, \tilde{z})| &= \left| \frac{1}{\partial_y \Phi} (g(t, \Phi, \partial_y \Phi \tilde{z}) + \frac{1}{2} \partial_y^2 \Phi |\tilde{z}|^2) \right| \\ &\leq \frac{1}{|\partial_y \Phi|} (a + b|\Phi| + c|\partial_y \Phi||\tilde{z}|) \\ &\quad + \frac{1}{|\partial_y \Phi|} (f(|\Phi|)|\partial_y \Phi|^2 |\tilde{z}|^2) + \frac{1}{2} \frac{|\partial_y^2 \Phi|}{|\partial_y \Phi|} |\tilde{z}|^2. \end{aligned}$$

Set

$$h(\tilde{y}) = \frac{1}{2} \frac{\partial_y^2 \Phi}{\partial_y \Phi} = \frac{1}{2} \partial_y \log |\partial_y \Phi|.$$

We use Proposition 11.11, in [12], Section 11, p. 289 to bound the flow Φ and its derivatives. A localization argument combined with the p non-explosion condition [12], Section 11, definition 11.1, p. 282 for RDE give the boundedness of the map $\frac{1}{\partial_y \Phi}$. Hence

$$\frac{1}{|\partial_y \Phi|} (a + b|\Phi| + c|\partial_y \Phi||\tilde{z}|) \leq \tilde{a} + \tilde{b}|\tilde{z}|,$$

and

$$\frac{1}{|\partial_y \Phi|} (f(|\Phi|)|\partial_y \Phi|^2 |\tilde{z}|^2) \leq \tilde{c} f(|\Phi|) |\tilde{z}|^2.$$

It follows that

$$\begin{aligned} \tilde{g}(t, \tilde{y}, \tilde{z}) &\leq \tilde{a} + \tilde{b}|\tilde{z}| + \tilde{c} f(|\Phi|) |\tilde{z}|^2 + h(\tilde{y}) |\tilde{z}|^2 \\ &= \tilde{a} + \tilde{b}|\tilde{z}| + \tilde{f}(|\tilde{y}|) |\tilde{z}|^2, \end{aligned}$$

where the constants \tilde{a} and \tilde{b} depend on C_G and $\|\eta\|_{p-var}$.

We are now ready to prove Theorem 1.

Proof of Theorem 1. Using properties of the function \tilde{g} demonstrated in Lemma 2 and Theorem 3.1 in [2] see also [3] there exists a solution $(\tilde{Y}, \tilde{Z}) \in \mathcal{S}^2 \times \mathcal{M}^2$ to the quadratic BSDE($\xi, \tilde{g}, 0, 0$). Define

$$Y_t := \Phi(t, \tilde{Y}_t) \quad \text{and} \quad Z_t := \partial_y \Phi(t, \tilde{Y}_t) \tilde{Z}_t, \quad t \in [0, T].$$

Since \tilde{Y} is semimartingale, and Φ a rough flow of \mathcal{C}^3 -diffeomorphism, denoting $\tilde{\theta}_t = (t, \tilde{Y}_t)$ we

obtain by Itô's formula

$$\begin{aligned}
Y_t &= \Phi(\tilde{\theta}_t) \\
&= \xi + \int_t^T \partial_y \Phi(\tilde{\theta}_s) \left(\frac{1}{\partial_y \Phi(\tilde{\theta}_s)} g(s, \Phi(\tilde{\theta}_s), \partial_y \Phi(\tilde{\theta}_s) \tilde{Z}_s) + \frac{1}{2} \partial_y^2 \Phi(\tilde{\theta}_s) |\tilde{Z}_s|^2 \right) ds \\
&\quad + \int_t^T G(\Phi(\tilde{\theta}_s)) d\eta_s - \frac{1}{2} \int_t^T \partial_y^2 \Phi(\tilde{\theta}_s) |\tilde{Z}_s|^2 ds - \int_t^T \partial_y \Phi(\tilde{\theta}_s) \tilde{Z}_s dW_s \\
&= \xi + \int_t^T G(\Phi(\tilde{\theta}_s)) d\eta_s + \int_t^T g(s, \Phi(\tilde{\theta}_s), \partial_y \Phi(\tilde{\theta}_s) \tilde{Z}_s) ds \\
&\quad + \frac{1}{2} \int_t^T \partial_y^2 \Phi(\tilde{\theta}_s) |\tilde{Z}_s|^2 ds - \int_t^T \partial_y \Phi(\tilde{\theta}_s) \tilde{Z}_s dW_s - \frac{1}{2} \int_t^T \partial_y^2 \Phi(\tilde{\theta}_s) |\tilde{Z}_s|^2 ds \\
&= \xi + \int_t^T g(s, Y_s, Z_s) ds + \int_t^T G(Y_s) d\eta_s - \int_t^T Z_s dW_s.
\end{aligned}$$

Since the rough flow Φ and its derivatives are bounded c.f. Proposition 11.11 in [12], we deduce that $(Y, Z) \in \mathcal{S}^2 \times \mathcal{M}^2$. The proof is now finished.

In the next Corollary we establish the uniqueness of solutions for a class of rough QBSDE whose generator g is of the form $f(y)|z|^2$ and f is merely measurable and integrable over the whole real line.

Corollary 1. *Let $\gamma > p \geq 1$, $\eta \in C_0^{p-var}([0, T], G^{[p]}(\mathbb{R}^d))$ and f a real-valued integrable function. Assume that (H3) is satisfied and $\xi \in L^2(\mathcal{F}_T)$. Let Φ be the rough flow (3.7). Then the quadratic BSDE($\xi, f(y)|z|^2, G, \eta$) has a unique solution in $\mathcal{S}^2 \times \mathcal{M}^2$.*

Proof. Putting $g(t, y, z) = f(y)|z|^2$, we have

$$\begin{aligned}
\tilde{g}(t, \tilde{y}, \tilde{z}) &:= \frac{1}{\partial_y \Phi} (g(t, \Phi, \partial_y \Phi \tilde{z}) + \frac{1}{2} \partial_y^2 \Phi |\tilde{z}|^2) \\
&= \frac{1}{\partial_y \Phi} f(\Phi) |\partial_y \Phi|^2 |\tilde{z}|^2 + \frac{1}{2} \frac{\partial_y^2 \Phi}{\partial_y \Phi} |\tilde{z}|^2 \\
&= f(\Phi) \partial_y \Phi |\tilde{z}|^2 + \frac{1}{2} \partial_y \log |\partial_y \Phi| |\tilde{z}|^2.
\end{aligned}$$

Using the boundedness of the derivative of the flow, the function $f(\Phi) \partial_y \Phi$ is integrable. Moreover we can write

$$\tilde{g}(t, \tilde{y}, \tilde{z}) = \tilde{f}(\tilde{y}) |\tilde{z}|^2,$$

where $\tilde{f} = f(\Phi) \partial_y \Phi + \frac{1}{2} \partial_y \log |\partial_y \Phi|$ is an integrable function. Now, by Theorem 3.1 (assertion (A)) in [2] see also [3], we get the existence and uniqueness of solution $(\tilde{Y}, \tilde{Z}) \in \mathcal{S}^2 \times \mathcal{M}^2$ for the quadratic BSDE($\xi, \tilde{f}(\tilde{y}) |\tilde{z}|^2, 0, 0$). More precisely we deduce the existence of solution $(Y, Z) \in \mathcal{S}^2 \times \mathcal{M}^2$ from the proof of Theorem 1 for the rough quadratic BSDE($\xi, f(y)|z|^2, G, \eta$). Its uniqueness follows from the uniqueness of (\tilde{Y}, \tilde{Z}) and the fact that the mapping

$$(Y, Z) := L(\tilde{Y}, \tilde{Z}) := \left(\Phi(\cdot, \tilde{Y}), \partial_y \Phi(\cdot, \tilde{Y}) \tilde{Z} \right)$$

is one to one.

3.2 Zvonkin transformation

The second result of this paper is to give another method to obtain existence and uniqueness results for a large class of QBSDE with rough drivers and square integrable terminal data. In many situations we can assume that the generator of the QBSDE is merely only measurable and integrable in y . Our idea is, when the generator is the form $a + b|y| + c|z| + f(y)z^2$, the quadratic part of this last can be eliminated by using Zvonkin transformation which allows us to deduce the existence of QBSDE($\xi, a + b|y| + c|z| + f(y)z^2, G, \eta$) from a BSDE of the form BSDE($\xi, a + b|y| + c|z|, G, \eta$).

We consider the following assumption and recall the Itô–Krylov formula in QBSDE given in ([2], Theorem 2.1, [3]).

(H4) There exists a positive stochastic process $\zeta_t \in L^1([0, T] \times \Omega)$ and a locally integrable function f such that, for every (t, ω, y, z)

$$|g(t, y, z)| \leq \zeta_t + |f(y)||z|^2.$$

Theorem 2. (*Itô–Krylov’s formula for BSDEs*).

Let ξ be an \mathcal{F}_T -measurable and square integrable random variable and assume that (H4) holds. Let the process (Y, Z) be a solution of the quadratic BSDE($\xi, g, 0, 0$) in $\mathcal{S}^2 \times \mathcal{L}^2$. Assume moreover that $\int_0^T |g(s, Y_s, Z_s)| ds$ is finite \mathbb{P} -almost surely. Then, for any function u belonging to $\mathcal{C}^1(\mathbb{R}) \cap \mathcal{W}_{1,loc}^2(\mathbb{R})$, we have

$$u(Y_t) = u(Y_0) + \int_0^t u'(Y_s) dY_s + \frac{1}{2} \int_0^t u''(Y_s) |Z_s|^2 ds.$$

The following Lemma is key element in our approach.

Lemma 3. *Let f belongs to $L^1(\mathbb{R})$. The function*

$$u(x) := \int_0^x \exp\left(2 \int_0^y f(t) dt\right) dy$$

satisfies the following properties

1. $u''(x) - 2f(x)u'(x) = 0$ and $u \in \mathcal{C}^1(\mathbb{R}) \cap \mathcal{W}_{1,loc}^2(\mathbb{R})$.
2. u is a one to one function from \mathbb{R} onto \mathbb{R} .
3. The inverse function u^{-1} belongs to $\mathcal{C}^1(\mathbb{R}) \cap \mathcal{W}_{1,loc}^2(\mathbb{R})$.
4. Both u and u^{-1} are quasi-isometries.
5. If in addition f is continuous then both u and u^{-1} are of \mathcal{C}^2 .

Proof. Using the fact that f is the distributional derivative of $x \rightarrow \int_0^x f(t) dt$, we obtain the statement 1. The 5th one is obvious. For the rest of the proof, we refer to [1], [2], [3] and [24] for more details.

The second result of this paper is stated in the following theorem

Theorem 3. *Let $\gamma > p \geq 1$ and $\eta \in C_0^{p-var}([0, T], G^{[p]}(\mathbb{R}^d))$. Assume $\xi \in L^2(\mathcal{F}_T)$ and (H3) hold. Let $f \in L^1(\mathbb{R})$ and u the corresponding function defined in Lemma 3. Then,*

(i) (Y, Z) is the unique solution in $\mathcal{S}^2 \times \mathcal{M}^2$ of the quadratic BSDE($\xi, f(y)z^2, G, \eta$) if and only if the process (\tilde{Y}, \tilde{Z}) defined as

$$\tilde{Y} := u(Y), \quad \text{and} \quad \tilde{Z} := u'(Y)Z,$$

is the unique solution in $\mathcal{S}^2 \times \mathcal{M}^2$ to the BSDE

$$\tilde{Y}_t = \tilde{\xi} + \int_t^T \tilde{G}(\tilde{Y}_s) d\eta_s - \int_t^T \tilde{Z}_s dW_s,$$

where $\tilde{G}(x) = u'^{-1}(x)G(u^{-1}(x))$ and $\tilde{\xi} := u(\xi)$.

(ii) For any $a, b, c \in \mathbb{R}$, the process (Y, Z) is a solution in $\mathcal{S}^2 \times \mathcal{M}^2$ of the quadratic BSDE($\xi, a + b|y| + c|z| + f(y)z^2, G, \eta$) if and only if the process (\tilde{Y}, \tilde{Z}) is a solution in $\mathcal{S}^2 \times \mathcal{M}^2$ to the BSDE

$$\tilde{Y}_t = \tilde{\xi} + \int_t^T \tilde{g}(s, \tilde{Y}_s, \tilde{Z}_s) ds + \int_t^T \tilde{G}(\tilde{Y}_s) d\eta_s - \int_t^T \tilde{Z}_s dW_s, \quad (3.8)$$

where $\tilde{g}(t, y, z) = u'^{-1}(y)(a + b|u^{-1}(y)|) + c|z|$.

Proof. (i). Let (Y, Z) be the unique solution of the rough quadratic BSDE($\xi, f(y)z^2, G, \eta$). Since $u \in \mathcal{C}^1(\mathbb{R}) \cap \mathcal{W}_{1,loc}^2(\mathbb{R})$, then by Itô–Krylov’s formula we have

$$\begin{aligned} u(Y_t) &= u(\xi) - \int_t^T u'(Y_s) dY_s - \frac{1}{2} \int_t^T u''(Y_s) Z_s^2 ds \\ &= u(\xi) + \int_t^T u'(Y_s) f(Y_s) Z_s^2 ds + \int_t^T u'(Y_s) G(Y_s) d\eta_s \\ &\quad - \int_t^T u'(Y_s) Z_s dW_s - \frac{1}{2} \int_t^T u''(Y_s) Z_s^2 ds \\ &= u(\xi) + \int_t^T \left(u'(Y_s) f(Y_s) - \frac{1}{2} u''(Y_s) \right) Z_s^2 ds \\ &\quad + \int_t^T u'(Y_s) G(Y_s) d\eta_s - \int_t^T u'(Y_s) Z_s dW_s \\ &= u(\xi) - \frac{1}{2} \int_t^T (u''(Y_s) - 2f(Y_s)u'(Y_s)) Z_s^2 ds \\ &\quad + \int_t^T u'(Y_s) G(Y_s) d\eta_s - \int_t^T u'(Y_s) Z_s dW_s. \end{aligned}$$

By the assertion 1 of the Lemma 3 we have

$$u(Y_t) = u(\xi) + \int_t^T u'(Y_s) G(Y_s) d\eta_s - \int_t^T u'(Y_s) Z_s dW_s.$$

We put,

$$\tilde{Y}_t := u(Y), \quad \tilde{Z} := u'(Y)Z, \quad \text{and} \quad \tilde{\xi} := u(\xi),$$

then (\tilde{Y}, \tilde{Z}) satisfies the BSDE

$$\tilde{Y}_t = \tilde{\xi} + \int_t^T \tilde{G}(\tilde{Y}_s) d\eta_s - \int_t^T \tilde{Z}_s dW_s,$$

where $\tilde{G}(x) = u'^{-1}(x)G(u^{-1}(x))$.

Since

$$|\tilde{\xi}| \leq |\xi| \exp\left(\|f\|_{L^1(\mathbb{R})}\right),$$

then

$$\tilde{\xi} \in L^2(\mathcal{F}_T).$$

The function u is Lipschitz then

$$|\tilde{Y}_t| = |u(Y_t)| \leq C|Y_t| + |u(0)|,$$

where C is a constant.

Hence

$$\tilde{Y} \in \mathcal{S}^2.$$

By the uniform boundedness of u' , we get

$$\tilde{Z} \in \mathcal{M}^2.$$

Since u' is uniformly bounded, the hypothesis (H3) suffices to obtain uniqueness of solutions by Corollary 1. Hence, the process (\tilde{Y}, \tilde{Z}) is the unique solution in $\mathcal{S}^2 \times \mathcal{M}^2$ of the BSDE $(\tilde{\xi}, 0, \tilde{G}, \eta)$.

Conversely, suppose that the process (\tilde{Y}, \tilde{Z}) is the unique solution in $\mathcal{S}^2 \times \mathcal{M}^2$ of the BSDE $(\tilde{\xi}, 0, \tilde{G}, \eta)$. Since $u^{-1} \in \mathcal{C}^1(\mathbb{R}) \cap \mathcal{W}_{1,loc}^2(\mathbb{R})$, we have again by Itô–Krylov’s formula

$$\begin{aligned} Y_t &= u^{-1}(\tilde{Y}_t) \\ &= u^{-1}(\tilde{Y}_T) - \int_t^T (u^{-1})'(\tilde{Y}_s) d\tilde{Y}_s - \frac{1}{2} \int_t^T (u^{-1})''(\tilde{Y}_s) \tilde{Z}_s^2 ds \\ &= \xi + \int_t^T (u^{-1})'(\tilde{Y}_s) \tilde{G}(\tilde{Y}_s) d\eta_s - \int_t^T (u^{-1})'(\tilde{Y}_s) \tilde{Z}_s dW_s \\ &\quad - \frac{1}{2} \int_t^T (u^{-1})''(\tilde{Y}_s) \tilde{Z}_s^2 ds. \end{aligned}$$

Calculus implies

$$\begin{aligned} \int_t^T (u^{-1})'(\tilde{Y}_s) \tilde{G}(\tilde{Y}_s) d\eta_s &= \int_t^T (u^{-1})'(\tilde{Y}_s) u'^{-1}(\tilde{Y}_s) G(u^{-1}(\tilde{Y}_s)) d\eta_s \\ &= \int_t^T (u^{-1})'(\tilde{Y}_s) u'(Y_s) G(Y_s) d\eta_s \\ &= \int_t^T \frac{1}{u'(Y_s)} u'(Y_s) G(Y_s) d\eta_s \\ &= \int_t^T G(Y_s) d\eta_s, \end{aligned}$$

$$\begin{aligned}
\int_t^T (u^{-1})'(\tilde{Y}_s) \tilde{Z}_s dW_s &= \int_t^T \frac{1}{u'(Y_s)} u'(Y_s) Z_s dW_s \\
&= \int_t^T Z_s dW_s,
\end{aligned}$$

and also

$$\frac{1}{2} \int_t^T (u^{-1})''(\tilde{Y}_s) \tilde{Z}_s^2 ds = \frac{1}{2} \int_t^T (u^{-1})''(\tilde{Y}_s) (u'(Y_s))^2 Z_s^2 ds.$$

Since

$$(u^{-1})''(\tilde{Y}_s) = \left(\frac{1}{u'(Y_s)} \right)' = -2 \frac{f(Y_s) u'(Y_s)}{(u'(Y_s))^2},$$

then

$$\frac{1}{2} \int_t^T (u^{-1})''(\tilde{Y}_s) \tilde{Z}_s^2 ds = \int_t^T f(Y_s) Z_s^2 ds.$$

Putting things together, we obtain

$$Y_t = \xi + \int_t^T f(Y_s) Z_s^2 ds + \int_t^T G(Y_s) d\eta_s - \int_t^T Z_s dW_s.$$

Moreover

$$|Y_t| = |u^{-1}(\tilde{Y}_t)| \leq C |\tilde{Y}_t| + |u^{-1}(0)|,$$

since the function u^{-1} is Lipschitz, therefore

$$Y \in \mathcal{S}^2.$$

Member that $Z_t = \frac{\tilde{Z}_t}{u'(u^{-1}(\tilde{Y}_t))}$, with the inequality

$$\begin{aligned}
\left| \frac{1}{u'(x)} \right| &\leq \exp \left(-2 \int_0^x f(t) dt \right) \\
&\leq \exp \left(2 \int_0^x |f(t)| dt \right) \\
&\leq \exp \left(2 \int_0^{|x|} |f(t)| dt \right) \\
&\leq \exp \left(2 \|f\|_{L^1(\mathbb{R})} \right),
\end{aligned}$$

one shows that Z belongs to \mathcal{M}^2 , which means that (Y, Z) is a solution in $\mathcal{S}^2 \times \mathcal{M}^2$ of the rough quadratic BSDE $(\xi, f(y)z^2, G, \eta)$. Its uniqueness follows from the uniqueness of (\tilde{Y}, \tilde{Z}) and the fact that the mapping

$$(Y, Z) := \mathcal{L}(\tilde{Y}, \tilde{Z}) := \left(u^{-1}(\tilde{Y}), \frac{\tilde{Z}}{u'(u^{-1}(\tilde{Y}))} \right)$$

is one to one.

(ii). The proof of this assertion is similar to that of (i), so the detail of the Itô–Krylov’s formula are omitted.

We only need to establish the existence of solutions to BSDE (3.8). Itô–Krylov’s formula applied to the function u shows that

$$\tilde{Y}_t = \tilde{\xi} + \int_t^T \tilde{g}(s, \tilde{Y}_s, \tilde{Z}_s) ds + \int_t^T \tilde{G}(\tilde{Y}_s) d\eta_s - \int_t^T \tilde{Z}_s dW_s, \quad (3.9)$$

where

$$\tilde{g}(t, y, z) = u'^{-1}(y)(a + b|u^{-1}(y)|) + c|z|.$$

The function \tilde{g} is continuous, and when we use the boundedness of u' and the Lipschitz property of u^{-1} , we get

$$|\tilde{g}(t, y, z)| \leq a + b|y| + c|z| \leq a + b|y| + c|z| + |f(y)||z|^2.$$

By Theorem 1 the BSDE (3.9) has a solution in $\mathcal{S}^2 \times \mathcal{M}^2$. We use the same technique developed in (i) and Itô–Krylov’s formula to get existence of solutions of BSDE($\xi, a + b|y| + c|z| + f(y)z^2, G, \eta$)

Remark 2. *In contrast to [11], our approaches cover the BSDE with linear growth (put $f = 0$).*

3.3 Examples of application

3.3.1 Connection to Backward doubly SDEs

We do the connection with the so-called backward doubly stochastic differential equations (BDSDE) introduced by Pardoux and Peng in [22]. We recall that on $\mathcal{C}([0, T], \mathbb{R}^d)$ there exists a unique Borel probability measure, is known as the d –dimensional Wiener measure, so that the coordinate function $B_t(\omega) = \omega_t$ defines a Brownian motion. To begin with, let $\Omega^1 = \mathcal{C}([0, T], \mathbb{R}^d)$ and $\Omega^2 = \mathcal{C}([0, T], \mathbb{R}^m)$ equipped respectively with Wiener measures \mathbb{P}^1 and \mathbb{P}^2 . Consider $\Omega = \Omega^1 \times \Omega^2$ on which we define the product measure $\mathbb{P} := \mathbb{P}^1 \otimes \mathbb{P}^2$. For $(\omega^1, \omega^2) \in \Omega$, we define $B(\omega^1, \omega^2) := \omega^1$. Analogously, we define $W(\omega^1, \omega^2) := \omega^2$. Hence B is a d –dimensional Brownian motion and W is an independent m –dimensional Brownian motion. Let $\mathcal{F}_t := \mathcal{F}_{t,T}^B \vee \mathcal{F}_{0,t}^W$, where $\mathcal{F}_{t,T}^B := \sigma(B_r : r \in [t, T]) \vee \mathcal{N}^1$, $\mathcal{F}_{0,t}^W := \sigma(W_r : r \in [0, t]) \vee \mathcal{N}^2$ and \mathcal{N}^i is set of \mathbb{P}^i –null sets, $i = 1, 2$. Note that the collection $(\mathcal{F}_t, t \in [0, T])$ is neither increasing nor decreasing, and it does not constitute a filtration.

Given ξ in $L^2(\mathcal{F}_T)$ Pardoux and Peng [22] considered the following BDSDE

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds + \int_t^T G(Y_s) \circ dB_s + \int_t^T Z_s dW_s, \quad 0 \leq t \leq T. \quad (3.10)$$

An \mathcal{F} –adapted process (Y, Z) is called a solution of the above BDSDE if $\mathbb{E}[\sup_{t \leq T} |Y_t|^2] < \infty$, $\mathbb{E}[\int_0^T |Z_s|^2 ds] < \infty$ and \mathbb{P} –a.s. (3.10) is satisfied for $0 \leq t \leq T$. Under appropriate (essentially Lipschitz) conditions on g and G they establish existence and uniqueness of a solution.

Note that in [22] Pardoux and Peng considered the equations (3.10) where the Stratonovich integral is actually a Backward Itô integral. But if G is smooth enough, the formulations are equivalent.

We are interested now by the connection of the BDSDEs and rough drivers. This motivates us to take $2 < p < 3$ and define the lift Brownian motion to a process with values in $\mathbb{R}^d \oplus so(d)$, where $so(d)$ denotes the space of anti-symmetric $d \times d$ –matrices.

Definition 3. (*Lévy’s area*). *Given a d –dimensional Brownian motion $B = (B^1, B^2, \dots, B^d)$, we define the d –dimensional Lévy area $A = (A^{i,j} : i, j \in \{1, \dots, d\})$ as the continuous process*

$$t \longrightarrow A_t^{i,j} = \frac{1}{2} \left(\int_0^t B_s^i dB_s^j - B_s^j dB_s^i \right).$$

We note that A_t takes values in $so(d)$. In the sequel, \exp denotes the exponential map from $\mathbb{R}^d \oplus so(d)$ to $G^2(\mathbb{R}^d)$. Set

$$G^2(\mathbb{R}^d) := \exp(\mathbb{R}^d \oplus so(d)) = \{(1, v, \frac{1}{2}v \otimes v + A); v \in \mathbb{R}^d \text{ and } A \in so(d)\}.$$

Definition 4. Let B and A denote a d -dimensional Brownian motion and its Lévy area process. The continuous $G^2(\mathbb{R}^d)$ -valued process \mathbf{B} , defined by

$$\mathbf{B}_t := \exp(B_t + A_t), \quad t \geq 0$$

is called *enhanced Brownian motion (EBM)*. \mathbf{B} is precisely d -dimensional Brownian motion enhanced with its iterated integrals in Stratonovich sense. It is in one to one correspondence with Brownian motion enhanced with Lévy's area.

The EBM \mathbf{B} has finite p -variation for $p \geq 2$. by setting $\mathbf{B} = 0$ on a \mathbb{P}^1 -null sets, we can say that \mathbf{B} belongs to $\mathcal{C}_0^{p-var}([0, T], G^2(\mathbb{R}^d))$. The EBM can be identified as a special case of left-invariant Brownian motion on the Lie group $G^2(\mathbb{R}^d)$. We refer to Section 13 in [12] for more details.

Theorem 4. Let $2 < p < 3$ and $\gamma > p$. We assume that (H1)–(H3) hold and $\xi \in L^2(\mathcal{F}_T)$. For every $\omega^1 \in \Omega^1$ Then the BSDE with rough driver for all $0 \leq t \leq T$

$$\begin{aligned} Y_t^{rp}(\omega^1, \cdot) &= \xi(\cdot) + \int_t^T g(s, Y_s^{rp}(\omega^1, \cdot), Z_s^{rp}(\omega^1, \cdot)) ds \\ &\quad + \int_t^T G(Y_s^{rp}(\omega^1, \cdot)) d\mathbf{B}_s(\omega^1) + \int_t^T Z_s^{rp}(\omega^1, \cdot) dW_s(\cdot). \end{aligned} \quad (3.11)$$

has a solution. In particular, if $g(t, y, z) = f(y)|z|^2$, where f is an integrable function, the uniqueness holds true for this equation. Moreover by (3.10) we have for \mathbb{P}^1 -a.e. ω^1 that \mathbb{P}^2 -a.s.

$$Y_t(\omega^1, \cdot) = Y_t^{rp}(\omega^1, \cdot), \quad t \leq T,$$

and

$$Z_t(\omega^1, \cdot) = Z_t^{rp}(\omega^1, \cdot), \quad dt \otimes \mathbb{P}^2\text{-a.s.}$$

Proof. As in the proof of Theorem 1, in the BDSDEs setting, we eliminate the integral corresponding to the Brownian motion B using the stochastic flow ϕ , defined as the unique solution of the stochastic differential equation in the Stratonovich sense

$$\phi(t, \omega^1; y) = y + \int_t^T G(\phi(s, \omega^1; y)) \circ dB_s(\omega^1).$$

Then ω^1 -wise, we construct the rough flow given by

$$\Phi(t, \omega^1; y) = y + \int_t^T G(\Phi(s, \omega^1; y)) d\mathbf{B}_s(\omega^1).$$

Therefore by Theorem 1 we obtain the result.

If $g(t, y, z) = f(y)|z|^2$, by Corollary 1 we get uniqueness of the solution (Y^{rp}, Z^{rp}) . By a classical result of rough path theory, we have for every $\omega^1 \in \Omega^1$

$$\Phi(\cdot, \omega^1; \cdot) = \phi(\cdot, \omega^1; \cdot).$$

Hence processes (Y, Z) and (Y^{rp}, Z^{rp}) satisfy the same BSDE. Therefore, we get the desired result by uniqueness.

3.3.2 Connection to fractional Brownian motion

We have seen that d -dimensional Brownian motion B can be enhanced to a stochastic process \mathbf{B} for which every realization is a geometric p -rough path, $p \in (2, 3)$. Recall that B is a continuous, centered Gaussian process with independent components (B^1, \dots, B^d) , whose law is fully determined by its covariance function

$$R(s, t) = \mathbb{E}(B_s \otimes B_t) = \text{diag}(s \wedge t, \dots, s \wedge t).$$

We note that this covariance function $R := R(s, t)$ has finite 1-variation in $2D$. More generally, consider a d -dimensional continuous, centered Gaussian process with independent components $X = (X_t^1, \dots, X_t^d : t \in [0, T])$. Again its law is fully determined by its covariance function

$$R(s, t) = \text{diag}(\mathbb{E}(X_s^1 X_t^1), \dots, \mathbb{E}(X_s^d X_t^d)), \quad s, t \in [0, T].$$

Let $p \in (3, 4)$. When the covariance function of X has finite ρ -variation for some $\rho \in [1, 2)$, Friz and Victoir ([12], Section 15, Theorem 5.33) have shown the existence of unique process \mathbf{X} , which lies in $\mathcal{C}_0^{p\text{-var}}([0, 1], G^3(\mathbb{R}^d))$, lifting the Gaussian process X for any $p > 2\rho$. This $G^3(\mathbb{R}^d)$ -valued process \mathbf{X} is called the enhanced Gaussian process and sample path realizations of \mathbf{X} are called Gaussian rough paths. Theorem 5.33 in [12] asserts in particular that d -dimensional Brownian motion can be lifted to an enhanced Gaussian process. Other example can be obtained by considering d -independents copies of fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$. The resulting \mathbb{R}^d -valued fBm can be lifted to an enhanced Gaussian process provided $H > \frac{1}{4}$. Recall that a d -dimensional fBm with Hurst parameter $H \in (0, 1)$ is a Gaussian process B^H

$$B_t^H := (B_t^{H,1}, \dots, B_t^{H,d}), \quad t \geq 0,$$

where $B^{H,1}, \dots, B^{H,d}$ are d independents centered Gaussian processes with covariance function

$$R(s, t) = \frac{1}{2} (s^{2H} + t^{2H} - |t - s|^{2H}), \quad (s, t) \in [0, +\infty]^2.$$

Let us consider the following stochastic flow defined as the solution of the Stochastic Differential Equation (SDEs) in the Stratonovich sense, driven by a fBm with Hurst parameter $H > \frac{1}{4}$

$$\phi(t, \omega^1; y) = y + \int_t^T G(\phi(s, \omega^1; y)) \circ dB_s^H(\omega^1).$$

If $H = \frac{1}{2}$, this equation corresponds to SDEs driven by Brownian motion in Stratonovich sense. When H is greater than $\frac{1}{2}$ existence and uniqueness of the solution are obtained by Zähle [23] and Nualart and Rascânu [20] and the references therein. In the case $H < \frac{1}{2}$, since fBm has α -Hölder continuous sample paths for any $\alpha < H$, it falls into the rough paths theory. When $H \neq \frac{1}{2}$ the fBm is neither a semimartingale nor a Markov process. Hence, a natural application of the rough path analysis is the stochastic calculus with respect to the fBm.

We consider $\mathcal{F}_t := \mathcal{F}_{t,T}^{B^H} \vee \mathcal{F}_{0,t}^W$, where $\mathcal{F}_{t,T}^{B^H} = \sigma(B_s^H : s \in [t, T]) \vee \mathcal{N}$ and \mathcal{N} is the set of \mathbb{P}^1 -negligible sets. The canonical processes on $\Omega = \Omega^1 \times \Omega^2$ are defined by $B^H(\omega^1, \omega^2) := \omega^1$ and $W(\omega^1, \omega^2) := \omega^2$. Hence B^H is a d -dimensional fBm.

Let $3 < p < 4$, $H > \frac{1}{4}$ be such that $Hp > 1$. In this setting Coutin and Qian [8], by using dyadic approximations showed the existence of a canonical geometric p -rough path \mathbf{B}^H

associated to the fBm B^H with Hurst parameter $H \in (\frac{1}{4}, \frac{1}{2})$. Setting $\mathbf{B}^H = 0$ on \mathcal{N} we assume that \mathbf{B}^H lies in $\mathcal{C}_0^{p-var}([0, T], G^3(\mathbb{R}^d))$. Consider the ω^1 -wise fractional rough flow

$$\Phi(t, \omega^1; y) = y + \int_t^T G(\Phi(s, \omega^1; y)) d\mathbf{B}_s^H(\omega^1). \quad (3.12)$$

Theorem 5. *Let $3 < p < 4$, $\gamma > p$ and $H > \frac{1}{4}$ be such that $Hp > 1$. We assume that (H1)–(H3) hold and $\xi \in L^2(\mathcal{F}_T)$. For every $\omega^1 \in \Omega^1$ the BSDE with rough driver for $0 \leq t \leq T$*

$$\begin{aligned} Y_t^{rp}(\omega^1, \cdot) &= \xi(\cdot) + \int_t^T g(s, Y_s^{rp}(\omega^1, \cdot), Z_s^{rp}(\omega^1, \cdot)) ds \\ &\quad + \int_t^T G(Y_s^{rp}(\omega^1, \cdot)) d\mathbf{B}_s^H(\omega^1) + \int_t^T Z_s^{rp}(\omega^1, \cdot) dW_s(\cdot), \end{aligned}$$

has a solution. Moreover the uniqueness holds true for $g(t, y, z) = f(y)|z|^2$ and f is an integrable function.

Proof. We use the rough flow (3.12), Theorem 1 and Corollary 1.

4 Probabilistic representation of rough PDEs

The aim of this section is to give a probabilistic representation of the following Quadratic PDE with rough path:

$$\begin{cases} du(t, x) + [\mathcal{L}u(t, x) + f(u(t, x))|u_x(t, x)\sigma(t, x)|^2] dt + G(u(t, x))d\eta_t = 0, \\ u(T, x) = \psi(x), \quad x \in \mathbb{R}. \end{cases} \quad (4.13)$$

where

$$\mathcal{L}u(t, x) := \frac{1}{2} (\sigma^2 u_{xx})(t, x) + (bu_x)(t, x).$$

Assume first that the equation (4.13) has a classical smooth solution. Let X be the unique solution to the following forward equation

$$X_t^{s,x} = x + \int_s^t b(r, X_r^{s,x}) dr + \int_s^t \sigma(r, X_r^{s,x}) dW_r, \quad t \in [s, T], \quad (4.14)$$

where the functions σ and b are given coefficients defined as follows: $\sigma : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, $b : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, $f : \mathbb{R} \rightarrow \mathbb{R}$, and $\psi : \mathbb{R} \rightarrow \mathbb{R}$.

such that:

(H4) σ, b are uniformly Lipschitz.

(H5) σ, b are of linear growth.

These condition insure existence and uniqueness of the equation (4.14).

Consider the following rough QBSDE

$$Y_t^{s,x} = \psi(X_T^{s,x}) + \int_t^T f(Y_r^{s,x}) |Z_r^{s,x}|^2 dr + \int_t^T G(Y_r^{s,x}) d\eta_r - \int_t^T Z_r^{s,x} dB_r \quad (4.15)$$

where f, G and ψ are given measurable functions such that.

(H6) f is continuous and integrable, G is Lipschitz and ψ is continuous and $|\psi(x)| \leq K(1 + |x|^p)$, $\forall p \geq 1$.

Applying Itô's formula to $u(T, X_T^{s,x})$ yields

$$\begin{aligned}
u(T, X_T^{s,x}) &= u(t, X_t^{s,x}) + \int_t^T (u_t + \mathcal{L}u)(r, X_r^{s,x}) dr \\
&\quad + \int_t^T (\sigma u_x)(r, X_r^{s,x}) dW_r \\
&= u(t, X_t^{s,x}) - \int_t^T f(u(r, X_r^{s,x})) |(u_x \sigma)(r, X_r^{s,x})|^2 dr \\
&\quad - \int_t^T G(u(r, X_r^{s,x})) d\eta_r + \int_t^T (\sigma u_x)(r, X_r^{s,x}) dW_r.
\end{aligned}$$

which means that $(u(r, X_r^{s,x}), (\sigma u_x)(r, X_r^{s,x}))_{s \leq r \leq T}$ is a solution to the (4.15).

The purpose of this section is to study the converse.

Given a solution to the (4.15) which is unique by our result in the previous section. We shall construct a viscosity solution to the rough quadratic PDE (4.13).

When the rough path η is replaced by a smooth path η , then (4.13) takes the form

$$\begin{cases} \partial_t u(t, x) + \mathcal{L}u(t, x) + f(u(t, x)) |(u_x \sigma)(t, x)|^2 + G(u(t, x)) \dot{\eta}_t = 0, \\ u(T, x) = \psi(x), \quad x \in \mathbb{R}, \end{cases} \quad (4.16)$$

We again consider the flows associated with a smooth path η (resp., rough η) defined in (3.4) (resp. in (3.7)).

Proposition 2. *Let G be a Lipschitz function on \mathbb{R} and η a given smooth path. Under the assumptions (H4)–(H6), $u(t, x) := Y_t^{t,x}$ is a viscosity solution to the PDE (4.16), where for every $(s, x) \in [0, T] \times \mathbb{R}^n$, the process $(Y^{s,x}, Z^{s,x})$ is the unique solution of QBSDE*

$$\begin{aligned}
Y_t^{s,x} &= \psi(X_T^{s,x}) + \int_t^T f(Y_r^{s,x}) |Z_r^{s,x}|^2 dr \\
&\quad + \int_t^T G(Y_r^{s,x}) d\eta_r - \int_t^T Z_r^{s,x} dB_r.
\end{aligned} \quad (4.17)$$

For an introduction to the theory of viscosity solutions, we refer the reader to [9]. To prove the existence of viscosity solutions, we need the following touching property, see [14].

Lemma 4. *Let $(\xi_t)_{0 \leq t \leq T}$ be a continuous adapted process such that*

$$d\xi_t = \beta_t dt + \alpha_t dW_t,$$

where β and α are continuous adapted processes such that $\beta, |\alpha|^2$ are integrable. If $\xi_t \geq 0$ a.s. for all t , then for all t ,

$$\begin{aligned} \mathbf{1}_{\{\xi_t=0\}} \alpha_t &= 0, \quad \text{a.s.}, \\ \mathbf{1}_{\{\xi_t=0\}} \beta_t &\geq 0, \quad \text{a.s.} \end{aligned}$$

Proof. First notice that

$$\forall t \in [s, T], \quad u(t, X_t^{s,x}) = Y_t^{s,x}. \quad (4.18)$$

This is readily seen from the Markov property of the diffusion process X and from the uniqueness of the solutions of the BSDE (4.17). Then $u(t, x) = Y_t^{t,x}$. Let $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^n)$. Let (t, x) be a local Maximum of $u - \varphi$. We suppose it global and equal to 0, that is

$$\varphi(t, x) = u(t, x) \quad \text{and} \quad \varphi(\bar{t}, \bar{x}) \geq u(\bar{t}, \bar{x}) \quad \text{for all } (\bar{t}, \bar{x}) \in [0, T] \times \mathbb{R}^n.$$

This and equality (4.18) imply that

$$\varphi(t, X_t^{s,x}) \geq Y_t^{s,x}.$$

We want to show that u is a viscosity supersolution of (4.16). Remember that $(Y^{s,x}, Z^{s,x})$ satisfy

$$\begin{aligned} Y_t^{s,x} &= Y_T^{s,x} + \int_t^T f(Y_r^{s,x}) |Z_r^{s,x}|^2 dr \\ &\quad + \int_t^T G(Y_r^{s,x}) \dot{\eta}_r dr - \int_t^T Z_r^{s,x} dW_r. \end{aligned}$$

We apply Itô's formula to the process $\varphi(t, X_t^{s,x})$, then we obtain

$$\begin{aligned} \varphi(T, X_T^{s,x}) &= \varphi(t, X_t^{s,x}) + \int_t^T (\varphi_t + \mathcal{L}\varphi)(r, X_r^{s,x}) dr \\ &\quad + \int_t^T (\varphi_x \sigma)(r, X_r^{s,x}) dW_r. \end{aligned}$$

As $\varphi(t, X_t^{s,x}) \geq Y_t^{s,x}$, the touching property gives for all t ,

$$\begin{aligned} &\mathbf{1}_{\{\varphi(t, X_t^{s,x}) = Y_t^{s,x}\}} \left[\varphi_t(t, X_t^{s,x}) + \mathcal{L}\varphi(t, X_t^{s,x}) \right. \\ &\quad \left. + f(Y_t^{s,x}) |Z_t^{s,x}|^2 + G(Y_t^{s,x}) \dot{\eta}_t \right] \geq 0, \quad \text{a.s.}, \end{aligned} \quad (4.19)$$

$$\mathbf{1}_{\{\varphi(t, X_t^{s,x}) = Y_t^{s,x}\}} \left[-Z_t^{s,x} + (\varphi_x \sigma)(t, X_t^{s,x}) \right] = 0, \quad \text{a.s.} \quad (4.20)$$

Or $\varphi(t, x) := \varphi(t, X_t^{t,x}) = Y_t^{t,x} := u(t, x)$ for $s = t$, then equation (4.20) gives $Z_t^{t,x} = (\varphi_x \sigma)(t, x)$ and equation (4.19) gives the expected result.

Lemma 5. *Let G be a Lipschitz on \mathbb{R} , η a given smooth path and ϕ the flow defined in (3.4). We assume that (H4)–(H6) hold and $(Y^{s,x}, Z^{s,x})$ is the unique solution of the QBSDE (4.17). Then $u(t, x) = Y_t^{t,x}$ is a viscosity solution to the PDE (4.16) if and only if $v(t, x) = \phi^{-1}(t, u(t, x))$ is a viscosity solution of the PDE*

$$\begin{cases} \partial_t v(t, x) + \mathcal{L}v(t, x) + \tilde{f}(t, v(t, x), v_x(t, x) \sigma(t, x)) = 0, \\ v(T, x) = \psi(x), \quad x \in \mathbb{R}, \end{cases}$$

where (in what follows the ϕ will be evaluated at (t, \tilde{y}))

$$\tilde{f}(t, \tilde{y}, \tilde{z}) := \frac{1}{\partial_y \phi} \left(f(\phi) |\partial_y \phi \tilde{z}|^2 + \frac{1}{2} \partial_y^2 \phi |\tilde{z}|^2 \right).$$

Proof. Suppose that u is a viscosity solution to the PDE (4.16). The function u is continuous, hence v is also continuous. We have

$$v(T, x) = \phi^{-1}(T, u(T, x)) = \phi^{-1}(T, \psi(x)) = \psi(x).$$

By Lemma 1 and Remark 1, the BSDE($\psi(X_T^{s,x}), \tilde{f}, 0, 0$) has a unique solution $\tilde{Y}_t^{s,x} = \phi^{-1}(t, Y_t^{s,x})$ and it is connected to the following PDE

$$\begin{cases} \partial_t \tilde{u}(t, x) + \mathcal{L}\tilde{u}(t, x) + \tilde{f}(t, \tilde{u}(t, x), \tilde{u}_x(t, x) \sigma(t, x)) = 0, \\ \tilde{u}(T, x) = \psi(x), \quad x \in \mathbb{R}, \end{cases}$$

which has a viscosity solution given by

$$\tilde{u}(t, x) := \tilde{Y}_t^{t, x} = \phi^{-1}(t, Y_t^{t, x}) = \phi^{-1}(t, u(t, x)) = v(t, x).$$

For the converse, we apply Itô's formula to the process $Y_t^{s, x} = \phi(t, \tilde{Y}_t^{s, x})$ and use the touching property.

Lemma 6. *Let $\gamma > p \geq 1$, $\eta \in \mathcal{C}_0^{p-var}([0, T], G^{[p]}(\mathbb{R}^d))$ and Φ the rough flow defined in (3.7). Under the assumption (H3), the function*

$$\tilde{f}(t, \tilde{y}, \tilde{z}) := \frac{1}{\partial_y \Phi} \left(f(\Phi) |\partial_y \Phi \tilde{z}|^2 + \frac{1}{2} \partial_y^2 \Phi |\tilde{z}|^2 \right).$$

is continuous and integrable.

Proof. The continuity of the function f is obvious. The integrability assumption can be verified using Corollary 1.

4.1 Main result

Theorem 6. *Let $\gamma > p \geq 1$, $\eta \in \mathcal{C}_0^{p-var}([0, T], G^{[p]}(\mathbb{R}^d))$ and Φ the rough flow defined in (3.7). Under the assumptions (H3)–(H6), $u(t, x) = Y_t^{t, x}$ is a viscosity solution of the following rough PDE*

$$\begin{cases} du(t, x) + [\mathcal{L}u(t, x) + f(u(t, x)) |u_x(t, x) \sigma(t, x)|^2] dt + G(u(t, x)) d\eta_t = 0, \\ u(T, x) = \psi(x), \quad x \in \mathbb{R}, \end{cases}$$

where $(Y^{s, x}, Z^{s, x})$ is the unique solution of the rough BSDE $(\psi(X_T^{s, x}), f(y)z^2, G, \eta)$.

Proof. The function $v(t, x) := \tilde{Y}_t^{t, x}$ is a viscosity solution to the PDE

$$\begin{cases} \partial_t v(t, x) + \mathcal{L}v(t, x) + \tilde{f}(t, v(t, x), v_x(t, x) \sigma(t, x)) = 0, \\ v(T, x) = \psi(x), \quad x \in \mathbb{R}, \end{cases}$$

where $\tilde{Y}^{s, x}$ is the unique solution of the BSDE $(\psi(X_T^{s, x}), \tilde{f}, 0, 0)$. Here \tilde{f} is the function defined in Lemma 6. Define

$$Y_t^{s, x} := \Phi(t, \tilde{Y}_t^{s, x}).$$

By Corollary 1 the process $(Y^{s, x}, Z^{s, x})$ is the unique solution of the rough quadratic BSDE $(\psi(X_T^{s, x}), f(y)z^2, G, \eta)$. Hence we obtain

$$u(t, x) := Y_t^{t, x} = \Phi(t, v(t, x)),$$

and we write formally

$$\begin{cases} du(t, x) + (\mathcal{L}u(t, x) + f(u(t, x)) |u_x(t, x) \sigma(t, x)|^2) dt + G(u(t, x)) d\eta_t = 0, \\ u(T, x) = \psi(x), \quad x \in \mathbb{R}. \end{cases}$$

Conclusion.

In this paper, we have studied existence and uniqueness of a class of quadratic BSDE with rough drivers when the terminal condition is a square integrable random variable. We have given some examples to recover BDSDEs and also class of quadratic BSDE perturbed by a fractional Brownian motion. In the Markovian setting we have reestablished a probabilistic representation of a viscosity solution of rough PDEs by means of the nonlinear Feymann-Kac formula.

Remark also that the extension to appropriate multidimensional cases are straightforward.

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